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Deformation of complex Finsler metrics

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Abstract

The aim of this paper is to describe the infinitesimal deformation (M, V) of a complex Finsler space family $\{(M, L_t)\}_{t\in\mathbb{R}}$ and to study some of its geometrical objects (metric tensor, non-linear connection, etc). In this circumstances the induced non-linear connection on (M, V) is defined. Moreover we have elaborate the inverse problem, the problem of the first order deformation of the metric. A special part is devoted to the study of particular cases of the perturbed metric.

1 Introduction

The problem of complex structure deformations on a differentiable manifold is one of interest, and in this direction, remarkable results have been obtained [7, 8]. Starting with a complex (integrable) manifold (M, J), a deformation of a certain integrable almost complex structure is studied by a power series expansion in a real parameter t of the linear operator J, so that the obtained manifold (M, J_t) must be an integrable complex one. The problem is difficult and involves algebraic geometry considerations. Generally, through a deformation of a complex manifold, the entire geometry (the complexificate tangent space, Hermitian metrics, linear connections, etc.) is modified considerably.

The present work is intended to approach a simpler problem. We will not deform the manifold M, so the holomorphic tangent bundle $T'M$ remains the same. Instead, we change the metrics which acts on $T'M$, metrics which originates from a complex Finsler metric (M, L). In this way we obtain a family of

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complex Finsler spaces (M, L_t). This problem is called the *deformation of the complex Finsler structures.* To the best of our knolidge, the deformation of the complex Finsler structures was studied only by T. Aikou in [2], where the infinitesimal deformation of the Einstein-Finsler structures on a holomorphic bundle E is approached. Obviously, in the case when $E = T'M$, a special non-linear connection exits, called Chern-Finsler, which will bring new contributions to the study.

This paper deals with three main sections. The fist reviews only the most necessary notions for the main part. The second one begins by considering a family of spaces (M, L_t) given by the complex Finsler metric F, and defines the *infinitesimal deformation* (M, V) of the $(M, L = F^2)$ space. Here we are dealing with some geometry elements (metric tensor, non-linear connection, etc.) of the complex Finsler space (M, V), (Theorem 2.2). This point of view sheds some new light on the rigidity of an infinitesimal deformation of a Finsler space (Proposition 2.1). In contrast to previous section, the last part considers (M, L) a complex Finsler space, and defines the first variation of it as the one parameter family $\{\tilde{L}_t = L + tV\}_t$, where V is a real valued function. Under this assumption, the question is when $(M, \tilde{L}_t$ determines a family of complex Finsler spaces (Theorem 3.1). This problem has been called by us the *first order deformation.* What is important to be mentioned here is the relation between the induced connection of the deformation and the Chern-Finsler one (Theorem 3.2, Proposition 3.2). The advantage of using this connections lies in the fact that the characterization of the special subclasses is simplified. We will emphasize in our study only the pure Hermitian, the Kähler, the Berwald and the generalized Berwald spaces (Propositions 3.3, 3.4, 3.5, 3.6). This research includes also the characterization of the projective relation between the complex Finsler metrics L and $\tilde{L}_t$ (Proposition 3.7).

Let M be a complex manifold of complex dimension n, where $(U, (z^k))$ is a local chart with complex coordinates (z^k), and $T'M$ is holomorphic tangent bundle where the fiber has the (η^k) components. From now on we take into consideration $(M, L = F^2)$ as a complex Finsler space, where $F : T'M \to \mathbb{R}^+$ is called the *complex Finsler function* if it satisfies the below conditions:

i) $L := F^2$ is smooth on $\widetilde{T'M} := T'M \setminus \{0\}$;

ii) $F(z, \eta) \geq 0$, equality holds if and only if $\eta = 0$;

iii) $F(z, \lambda\eta) = |\lambda| F(z, \eta)$ for $\lambda \in \mathbb{C}$;

iv) the following Hermitian matrix $g_{j\bar{k}}(z, \eta)$, with

$$g_{j\bar{k}} = \frac{\partial^2 L}{\partial \eta^j \partial \bar{\eta}^k} \tag{1}$$

is positive definite on $\widetilde{T'M}$, and it is called *the fundamental metric tensor* of the space.

On (M, F) we consider the Chern-Finsler complex non-linear connection, (briefly (*c.n.c.*)) with the local coefficients $N_k^j = g^{\bar{m}j} \frac{g_{l\bar{m}}}{\partial z^k} \eta^l$. Consequently, the horizontal distribution $HT'M$ associated to Chern-Finsler (*c.n.c.*) will be generated by $\delta_k := \frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N_k^m \frac{\partial}{\partial \eta^m}$. Therefor, we obtain the local coefficients of the Chern-Finsler linear connection $D\Gamma = (N_j^i, L_{jk}^i, C_{jk}^i, 0, 0)$:

$$L_{kl}^j = g^{\bar{m}j} \delta_l g_{k\bar{m}}, \quad C_{kl}^j = g^{\bar{m}j} \dot{\partial}_l g_{k\bar{m}}. \tag{2}$$

The connection form and the curvature form is considered as follows:

$$\omega_k^j = L_{kl}^j dz^l + C_{kl}^j \delta\eta^l, \qquad \Omega_k^j = d''\omega_k^j. \tag{3}$$

Hence the Chern-Finsler (*c.l.c.*) is of type $(1, 0)$ for any $Z \in A^0(T'M)$ we have the decomposition $D = D' + D''$, with

$$D'Z = (d'Z^i + Z^m \omega_m^i) \otimes s_i, \quad D''Z = d''Z, \tag{4}$$

where $d'Z = \delta_m Z^m \mathrm{d}z^m + \dot{\partial}_m Z^m \delta\eta^m$ and $d''Z = \delta_{\bar{m}} Z^{\bar{m}} \mathrm{d}\bar{z}^m + \dot{\partial}_{\bar{m}} Z^{\bar{m}} \delta\bar{\eta}^m$.

In [1, 6] the notions of *weakly Kähler* space were introduced and studied, i.e. $g_{i\bar{l}} T_{jk}^i \eta^j \bar{\eta}^l = 0$, and of *Kähler* space, i.e. $T_{jk}^i \eta^j = 0$, where $T_{jk}^i = L_{jk}^i - L_{kj}^i$. If we work under the assumption that $g_{i\bar{j}} = g_{i\bar{j}}(z)$, then the space is called *purely Hermitian,* and the notions of weakly Kähler and Kähler coincides.

The Chern-Finsler (*c.n.c.*) in general doesn't derive from a complex spray, but always determines one with the local coefficients $G^i = \frac{1}{2} N_j^i \eta^j$. On the other hand, from G^i is obtained a (*c.n.c.*) through $\overset{c}{N_j^i} = 2\dot{\partial}_j G^i$, called *canonical* in [9]. This connection coincide with the Chern-Finsler one if and only if the metric is Kähler.

The Finsler space (M, L) is *generalized Berwald* if the coefficients G^i are holomorphic functions, i.e. $\dot{\partial}_{\bar{j}} G^i = 0$, ([3, 5]). A generalized Berwald space (M, L) which is also Kähler is called a *complex Berwald* space ([3]). The weakly Kähler form is given by $\theta^{*k} := g^{\bar{m}k} g_{h\bar{l}} T_{\bar{j}\bar{m}}^{\bar{l}} \eta^h \bar{\eta}^j$. A space with vanishing θ^{*k} becomes weakly Kähler.

Let $\tilde{L}$ be an other complex Finsler metric on M. Abate and Patrizio introduced in [1] the projective relation of two complex Finsler metrics, which means that the metrics L and $\tilde{L}$ on the manifold M have the same geodesics as set of points.

In [4] one can find necessary and sufficient conditions for projectively related complex Finsler metrics:

Theorem 1.1 ([4]). *Let L and $\tilde{L}$ be complex Finsler metrics on the manifold M. Then L and $\tilde{L}$ are projectively related if and only if there is a smooth function P in $T'M$ with complex values, such as*

$$\tilde{G}^i = G^i + Q^i + P\eta^i, \quad i = 1, \ldots, n,$$

where $Q^i := \frac{1}{2}(\tilde{\theta}^{*i} - \theta^{*i})$.

The following result is a complex version of Rapcsák's theorem.

Theorem 1.2 ([4]). *Let L and $\tilde{L}$ be complex Finsler metrics on the manifold M. Then L and $\tilde{L}$ are projectively related if and only if*

$$\begin{aligned} &\dot{\partial}_{\bar{r}}(\delta_k \tilde{L})\eta^k + 2(\dot{\partial}_{\bar{r}} G^l)(\dot{\partial}_l \tilde{L}) = \frac{1}{\tilde{L}}(\delta_k \tilde{L})\eta^k(\dot{\partial}_{\bar{r}} \tilde{L}); \\ &Q^r = -\frac{1}{2\tilde{L}}\theta^{*l}(\dot{\partial}_l \tilde{L})\eta^r; \\ &P = \frac{1}{2\tilde{L}}[(\delta_k \tilde{L})\eta^k + \theta^{*i}(\dot{\partial}_i \tilde{L})], \end{aligned} \tag{5}$$

$(r = 1, \ldots, n)$, *and the projective change is* $\tilde{G}^i = G^i + \frac{1}{2\tilde{L}}(\delta_k \tilde{L})\eta^k \eta^i$.

2 Infinitesimal deformations of Finsler structures

Let (M, L) be a complex Finsler space, with the fundamental tensor $g_{j\bar{k}}(z, \eta)$. We consider a $1-$parameter family of complex Finsler spaces $\{(M, L_t)\}_{t \in \mathbb{R}}$, where for each $t \in \mathbb{R}$ the functions $L_t(z, \eta)$ verifies the conditions i)-iv) on the $T'M$ holomorphic tangent bundle, and the metric tensors are
$g_{i\bar{j}}(t) := g_{i\bar{j}}(z, \eta, t)$ similar to (1). Suppose that for $t = 0$ we have $L_0 = L$. For this family of complex Finsler spaces we can consider a tangent vector:

$$V := \left(\frac{\partial L_t}{\partial t}\right)_{t=0} \tag{1}$$

called the *infinitesimal deformation* induced by the L_t family. Its components in respect with an orthonormal frame $\{\delta_k, \dot{\partial}_k, \delta_{\bar{k}}, \dot{\partial}_{\bar{k}}\}$ are given by:

$$v_{j\bar{k}} := \left(\frac{\partial g_{j\bar{k}}(t)}{\partial t}\right)_{t=0} \tag{2}$$

Since L_t are complex Finsler functions, we can deduce immediately that the V function is also smooth on $\widetilde{T'M}$, positive definite and homogeneous. However, this doesn't mean that the space (M, V) is also a complex Finsler one. For this purpose the function V needs to verify the following conditions:

Theorem 2.1. *Let (M, L) be a complex Finsler space with its infinitesimal deformation V defined in (1). If the V function satisfies the below properties:*

i) $V(z, \eta) \geq 0$, the equality holds if and only if $\eta = 0$;

ii) the matrix $v_{j\bar{k}} := \left(\frac{\partial g_{j\bar{k}}(t)}{\partial t}\right)_{t=0}$ is positive definite,

then the (M, V) will be a complex Finsler space, with the metric tensor $v_{j\bar{k}}$.

Remark 2.1. *The inverse of $v_{j\bar{k}}$ is $v^{\bar{k}m} := \frac{\partial g^{\bar{k}m}(t)}{\partial t}|_{t=0}$.*

We suppose that (M, V) is hereinafter a complex Finsler space. Between the metric tensors of (M, L) and (M, V) we have:

Lemma 2.1. *Between the tensors $g_{j\bar{k}}$ from (1) and $v_{j\bar{k}}$ from (2) we have:*

$$v_{j\bar{k}} g^{\bar{k}i} + g_{j\bar{k}} v^{\bar{k}i} = 0. \tag{3}$$

Proof. Our proof start with the observation that in (M, L) and in (M, L_t) it takes place:

$$g_{j\bar{k}} g^{\bar{k}m} = \delta_j^m \;\Rightarrow\; g_{j\bar{k}}(t) g^{\bar{k}m}(t) = \delta_j^m.$$

After the differentiation in respect with respect to t at $t = 0$, and we obtain:

$$g^{\bar{k}m} \frac{\partial g_{j\bar{k}(t)}}{\partial t}|_{t=0} + g_{j\bar{k}} \frac{\partial g^{\bar{k}m}(t)}{\partial t}|_{t=0} = 0 \;\Leftrightarrow v_{j\bar{k}} g^{\bar{k}m} + g_{j\bar{k}} v^{\bar{k}m} = 0.$$

□

We are interested in finding new non-linear connections in (M, V), and to establishing relations between them.

Theorem 2.2. *Let (M, L) be a complex Finsler space with his infinitesimal deformation (M, V). The functions $\overset{V}{N}{}_j^k := \frac{\partial N_j^k(t)}{\partial t}|_{t=0}$ are local coefficient of a (c.n.c.) in (M, V), called the* induced non-linear connection of the deformation.

Proof. According to [9] p.35, the functions $\overset{V}{N}{}_j^k$ are coefficients of a (*c.n.c.*), if the associated adapted frame $\overset{V}{\delta}_j := \partial_j - \frac{\partial N_j^k(t)}{\partial t}|_{t=0} \dot{\partial}_k$ simply changes with the matrix $\left(\frac{\partial z'^k}{\partial z^m}\right)$ on $T'M$, $\left(z^k, \eta^k\right) \to \left(z'^k, \eta'^k\right)$.

As $N_k^j(t)$ is the Chern-Finsler (*c.n.c.*) in (M, L_t), we have that his adapted frame $\delta_j := \partial_j - N_j^k(t) \dot{\partial}_k$ changes with the matrix $\left(\frac{\partial z'^k}{\partial z^m}\right)$. After a differentiation in respect to t, the same rule is preserved by the elements of the adapted frame, i.e. $\overset{V}{\delta}_j = \frac{\partial z'^k}{\partial z^j} \overset{V}{\delta'}_k$. □

On the other side, on (M,V) we can consider the Chern-Finsler $(c.n.c)$: $\overset{CFV}{N_j^k} = v^{\bar{l}k}\frac{\partial v_{p\bar{l}}}{\partial z^j}\eta^p$. We wish to investigate the link between this two $(c.n.c)$ on (M,V). For this aim, we recall that $N_k^j(t) = g^{\bar{m}j}(t)\frac{\partial g_{l\bar{m}}(t)}{\partial z^k}\eta^l$ and $N_k^j = N_k^j(0)$, and we explicit:

$$\begin{aligned}\frac{\partial v_{p\bar{l}}}{\partial z^j}\eta^p &= \frac{\partial}{\partial t}\left(\frac{\partial g_{p\bar{l}}(t)}{\partial z^j}\eta^p\right)_{t=0} = \frac{\partial}{\partial t}\left(g_{i\bar{l}}(t)N_j^i(t)\right)_{t=0} = \\ &= \frac{\partial g_{i\bar{l}}(t)}{\partial t}|_{t=0}N_j^i + g_{i\bar{l}}\frac{\partial N_j^i(t)}{\partial t}|_{t=0} = v_{i\bar{l}}N_j^i + g_{i\bar{l}}\frac{\partial N_j^i(t)}{\partial t}|_{t=0}.\end{aligned} \tag{4}$$

Applying this expression, we deduce a relation between the induced $(c.n.c.)$ on (M,V) and the Chern-Finsler one on (M,L):

$$\overset{V}{N_j^m} = -g^{\bar{l}m}v_{i\bar{l}}N_j^i + g^{\bar{l}m}\frac{\partial v_{p\bar{l}}}{\partial z^j}\eta^p. \tag{5}$$

The last formula can be processed in $\overset{V}{N_j^m} = v^{\bar{l}m}g_{i\bar{l}}g^{\bar{p}i}\frac{\partial g_{s\bar{p}}}{\partial z^j}\eta^s + g^{\bar{l}m}\frac{\partial v_{p\bar{l}}}{\partial z^j}\eta^p$, by using (3) can be written as:

$$\overset{V}{N_j^m} = v^{\bar{p}m}\frac{\partial g_{s\bar{p}}}{\partial z^j}\eta^s + g^{\bar{l}m}\frac{\partial v_{p\bar{l}}}{\partial z^j}\eta^p.$$

From (3) we obtaine $g^{\bar{l}m} = -v^{\bar{l}j}g_{j\bar{k}}v^{\bar{k}m}$. Replacing this in (5) we proved:

Theorem 2.3. *Let N be the Chern-Finsler $(c.n.c)$ in (M,L), let $\overset{V}{N}$ be the induced connection and let $\overset{VCF}{N}$ be the Chern-Finsler one on the infinitesimal deformation (M,V) of (M,L). Between this local coefficients occurs the following relation:*

$$\overset{V}{N_j^m} = g_{i\bar{l}}v^{\bar{l}m}(N_j^i - \overset{CFV}{N_j^i}). \tag{6}$$

In the next part we will look more closely at the linear connection of the infinitesimal deformation (M,V). Let D_t the Chern-Finsler linear connection of the 1-parameter family $\{(M,L_t)\}_{t\in\mathbb{R}}$ of the complex Finsler structure (M,L). We denote with $D_0 := D$, where D is the Chern-Finsler connection of the (M,L) space, and with D_t the Chern-Finsler connection of (M,L_t). Since the Chern-Finsler connection D_t is metric in respect to $g_{j\bar{k}}(t)$, i.e.

$$D_t g_{j\bar{k}}(t) = 0 \quad \overset{(4),(2),(3)}{\Leftrightarrow} \quad d'_t g_{j\bar{k}}(t) - \omega_j^m(t)g_{m\bar{k}}(t) = 0,$$

then the connection form $\omega(t)$ of D_t will be determined by

$$g_{i\bar{k}}(t)\omega_j^i(t) = d'_t g_{j\bar{k}}(t). \tag{7}$$

Now we consider a linear connection of (1,0)-type on the complex Finsler space (M,V), with the connection form

$$\overset{V}{\omega}{}_j^i = \left(\frac{\partial \omega_j^i(t)}{\partial t}\right)_{t=0}.$$

From the fact that $\omega_j^i(t)$ is a connection form on (M,L_t), immediately results that $\overset{V}{\omega}{}_j^i$ is also a connection form on (M,V), which will be called the *connection form of the infinitesimal deformation* of the Chern-Finsler connection.

To give the explicit form for the infinitesimal deformation of the Chern-Finsler connection $\omega(t) = D'_t + D''_t$ we differentiate (7) in respect to t at $t=0$:

$$g_{i\bar{k}}\left(\frac{\partial \omega_j^i(t)}{\partial t}\right)_{t=0} = \frac{\partial}{\partial t}\left(d'_t g_{j\bar{k}}(t)\right)_{t=0} - v_{i\bar{k}}\omega_j^i. \tag{8}$$

Let $\{\delta_k^t := \partial_k - N_k^j(t)\dot{\partial}_j, \dot{\partial}_k\}$ be the adapted base of the Chern-Finsler (*c.n.c.*) on $T'M$ for the family of functions L_t, and the dual basis $\{dz^k, \delta^t\eta^k := d\eta^k + N_j^k(t)dz^j\}$. We continue to develop independently the term $\frac{\partial}{\partial t}\left(d'_t g_{j\bar{k}}(t)\right)_{t=0}$ from (8):

$$\begin{aligned}
\frac{\partial}{\partial t}\left(d'_t g_{j\bar{k}}(t)\right)_{t=0} &= \frac{\partial}{\partial t}(\delta_m^t g_{j\bar{k}}(t)\mathrm{d}z^m + \dot{\partial}_m g_{j\bar{k}}(t)\delta^t\eta^m)_{t=0} \\
&= \frac{\partial}{\partial t}(\partial_m g_{j\bar{k}}(t)\mathrm{d}z^m - N_m^p(t)\dot{\partial}_p g_{j\bar{k}}(t)\mathrm{d}z^m \\
&\quad + \dot{\partial}_m g_{j\bar{k}}(t)\mathrm{d}\eta^m + \dot{\partial}_m g_{j\bar{k}}(t)N_p^m(t)\mathrm{d}z^p)_{t=0} \\
&= \frac{\partial}{\partial t}(\partial_m g_{j\bar{k}}(t)\mathrm{d}z^m + \dot{\partial}_m g_{j\bar{k}}(t)\mathrm{d}\eta^m)_{t=0} \\
&= \partial_m v_{j\bar{k}}\mathrm{d}z^m + \dot{\partial}_m v_{j\bar{k}}\mathrm{d}\eta^m \\
&= \partial_m v_{j\bar{k}}\mathrm{d}z^m + \dot{\partial}_m v_{j\bar{k}}\mathrm{d}\eta^m - N_m^p\dot{\partial}_p v_{j\bar{k}}\mathrm{d}z^m + N_p^m\dot{\partial}_m v_{j\bar{k}}\mathrm{d}z^p \\
&= \delta_m v_{j\bar{k}}\mathrm{d}z^m + \dot{\partial}_m v_{j\bar{k}}\delta\eta^m = d'v_{j\bar{k}}.
\end{aligned}$$

So (8) becomes

$$g_{i\bar{k}}\left(\frac{\partial \omega_j^i(t)}{\partial t}\right)_{t=0} = d'v_{j\bar{k}} - v_{i\bar{k}}\omega_j^i = D'v_{j\bar{k}}. \tag{9}$$

Contracted with $g^{\bar{k}m}$, and using $D'g_{j\bar{k}} = 0$ and $v_k^j := g^{\bar{m}j}v_{k\bar{m}}$, from (9) is obtained

$$\left(\frac{\partial \omega_j^m(t)}{\partial t}\right)_{t=0} = D'v_j^m.$$

We can now formulate an important result about the linear connection $\left(\frac{\partial D_t}{\partial t}\right)_{t=0}$ in (M, V).

Proposition 2.1. *Let L_t be the 1-parameter family of the complex Finsler metrics on $T'M$ with the infinitesimal deformation V. The infinitesimal deformation $\left(\frac{\partial D_t}{\partial t}\right)_{t=0}$ of the Chern-Finsler connection D is zero if and only if $D'V = 0$.*

Now we are able to write the non-zero coefficients of the infinitesimal deformation of the connection D_t:

$$\overset{V}{L}{}_{jk}^{m} := \left(\frac{\partial L_{jk}^i(t)}{\partial t}\right)_{t=0} = v^{\bar{m}i}\delta_k g_{j\bar{m}} + g^{\bar{m}i}\delta_k v_{j\bar{m}} - g^{\bar{m}i}\dot{\partial}_p g_{j\bar{m}} \overset{V}{N}{}_k^p$$

$$\overset{V}{C}{}_{jk}^{m} := \left(\frac{\partial C_{jk}^i(t)}{\partial t}\right)_{t=0} = v^{\bar{m}i}\dot{\partial}_k g_{j\bar{m}} + g^{\bar{m}i}\dot{\partial} v_{j\bar{m}},$$

where $\left(L_{jk}^m, C_{jk}^m\right)$ are given in (2).

Using the definition of the curvature and the curvature form given in (3), we obtain the expression of the *infinitesimal deformation of the curvature* associated with the Chern-Finsler connection:

$$\left(\frac{\partial \Omega_j^i(t)}{\partial t}\right)_{t=0} = D''\left(D'v_j'^i\right).$$

Obviously on the (M, V) space we can also consider the Chern-Finsler connection associated to the metric $v_{j\bar{k}}$, i.e. $\overset{CFV}{L}{}_{jk}^{m} = v^{\bar{m}j}\overset{CFV}{\delta_l} v_{k\bar{m}}$, $\overset{CFV}{C}{}_{kl}^{j} = v^{\bar{m}j}\dot{\partial}_l v_{k\bar{m}}$. The link between this connection with the connection of the infinitesimal deformation can be achieved by a trivial calculus, without interest.

3 First order deformation of a complex Finsler metric

Until now we have considered the complex Finsler space (M, L) and the family of complex Finsler spaces (M, L_t) which have defined the infinitesimal deformation V, for which we have presumed that satisfies the axioms of a complex Finsler function.

In the following, we treat the problem inversely. We consider the complex Finsler space (M, L) and $V : T'M \to \mathbb{R}^+$ a given function.

We define the family of functions $\tilde{L}_t : T'M \to \mathbb{R}^+$, $\forall t \in \mathbb{R}$, called *the first order deformation of* L :

$$\tilde{L}_t := L + tV, \quad \forall t \in \mathbb{R}.$$

Obviously, from (1), follows that V is an infinitesimal deformation (of first order) of $\tilde{L}_t$.

We search for the conditions under which $(M, \tilde{L}_t)$ are complex Finsler spaces. To achieve this, we must verify the four conditions from the definition of a complex Finsler function for $\tilde{L}_t$.

The ii) condition is equivalent with

$$\tilde{L}_t \geq 0 \;\Leftrightarrow\; L + tV \geq 0 \;\Leftrightarrow\; tV \geq -L, \forall t \in \mathbb{R}.$$

We verify the equality firstly from the converse, namely it assumes that $\eta = 0$. In this way we obtain

$$\tilde{L}_t(z, 0) = L(z, 0) + tV(z, 0) = tV(z, 0) = t\,\frac{\partial L_t(z, 0)}{\partial t}|_{t=0}. \tag{10}$$

This relation is not vanishing for all $t \in \mathbb{R},\ z \in M$. For example, if $L_t(z, 0) = zt$, then $\frac{\partial L_t(z,0)}{\partial t}|_{t=0} = z$. So, we must impose the condition $V(z, 0) = 0$, $\forall\ z \in T'M$. The function $\tilde{L}_t$ will be (1,1)-homogeneous if and only if V will be homogeneous of the same type. And so, we have:

Theorem 3.1. *The space $(M, \tilde{L}_t)$ with $\tilde{L}_t$ defined in (10) is a complex Finsler space if and only if*

i) the first order infinitesimal deformation V is a complex Finsler function,

ii) $tV \geq -L,\ \forall (z, \eta) \in T'M,\ t \in \mathbb{R}$,

iii) t is sufficiently small, so that he metric $\tilde{L}_t$ remains positive definite,

iv) the fundamental tensor $\tilde{g}_{j\bar{k}}(z, \eta, t)$ is positive definite, where

$$\tilde{g}_{j\bar{k}}(t) = \frac{\partial^2 \tilde{L}_t}{\partial \eta^j \partial \bar{\eta}^k} = g_{j\bar{k}}(z, \eta) + t v_{j\bar{k}}(z, \eta). \tag{11}$$

We assume that $(M, \tilde{L}_t)$ is hereinafter a complex Finsler space.

To study the geometrical objects of $(M, \tilde{L}_t)$ we need the inverse matrix of $(\tilde{g}_{j\bar{k}})$.

Proposition 3.1. *Let $(M, \tilde{L}_t)$ a complex Finsler space. The inverse of the fundamental metric tensor $\tilde{g}_{j\bar{k}}(t)$ from (11) is $\tilde{g}^{\bar{k}m}(z, \eta, t)$ with*

$$\tilde{g}^{\bar{k}m}(t) = \frac{1}{1+t^2} g^{\bar{k}m}(z, \eta) + \frac{t}{1+t^2} v^{\bar{k}m}(z, \eta). \tag{12}$$

Proof. The proof is made trough direct calculus, with the help of Lemma 2.1. □

In the study of the family of spaces $\{(M, \tilde{L}_t)\}_{t \in \mathbb{R}}$ an investigation of the non-linear connections is indispensable. On $(M, \tilde{L}_t)$ the Chern-Finsler (*c.n.c.*) has the following form

$$\begin{aligned}
\tilde{N}^i_j(t) &= \tilde{g}^{\bar{m}i}(t) \frac{\partial \tilde{g}_{p\bar{m}}(t)}{\partial z^j} \eta^p \\
&= \tilde{g}^{\bar{m}i}(t) \left(g_{\bar{p}m} N^p_j + t v_{\bar{p}m} \overset{CFV}{N^p_j} \right) \\
&= \left(\frac{1}{1+t^2} g^{\bar{m}i} + \frac{t}{1+t^2} v^{\bar{m}i} \right) \frac{\partial}{\partial z^j} (g_{p\bar{m}} + t v_{p\bar{m}}) \eta^p \\
&= \frac{1}{1+t^2} N^i_j + \frac{t}{1+t^2} \left(g^{\bar{m}i} \frac{\partial v_{p\bar{m}}}{\partial z^j} \eta^p + v^{\bar{m}i} \frac{\partial g_{p\bar{m}}}{\partial z^j} \eta^p \right) + \frac{t^2}{1+t^2} v^{\bar{m}i} \frac{\partial v_{p\bar{m}}}{\partial z^j} \eta^p \\
&= \frac{1}{1+t^2} N^i_j + \frac{t}{1+t^2} \frac{\partial N^i_j(t)}{\partial t}|_{t=0} + \frac{t^2}{1+t^2} v^{\bar{m}i} \frac{\partial v_{p\bar{m}}}{\partial z^j} \eta^p \qquad (13) \\
&= \frac{1}{1+t^2} N^i_j + \frac{t}{1+t^2} \overset{V}{N^i_j} + \frac{t^2}{1+t^2} \overset{CFV}{N^i_j} .
\end{aligned}$$

Theorem 3.2. *Let $(M, \tilde{L}_t)$ be a complex Finsler space. The complex non-linear connection Chern-Finsler $\tilde{N}^i_j(z, \eta, t)$ on $(M, \tilde{L}_t)$ is*

$$\tilde{N}^i_j(t) = N^i_j + \frac{t}{1+t^2} \frac{\partial N^p_j(t)}{\partial t}|_{t=0} \left(\delta^i_p - t v^i_p \right), \quad \forall\, t \in \mathbb{R}, \tag{14}$$

where N^i_j are the local coefficients of the Chern-Finsler (c.n.c.) on (M, L), and $v^i_p := v_{p\bar{m}} g^{\bar{m}i}$.

Proof. The demonstration is made with direct computations using the formula (13). After a differentiation with respect to t at $t = 0$ of $N^i_j(t) = g^{\bar{m}i}(t) \partial_j g_{p\bar{m}}(t) \eta^p$ and after a contraction with $g_{i\bar{l}}$ is obtained:

$$\frac{\partial(\partial_j g_{p\bar{l}})}{\partial t}|_{t=0} \eta^p = g_{i\bar{l}} \frac{\partial N^i_j(t)}{\partial t}|_{t=0} - g_{i\bar{l}} v^{\bar{m}i} \partial_j g_{p\bar{m}} \eta^p.$$

But $\partial_j g_{p\bar{m}}\eta^p = g_{r\bar{m}}N^r_j$ and $\frac{\partial(\partial_j g_{p\bar{l}})}{\partial t}|_{t=0} = \partial_j v_{p\bar{l}}$, and using (3) is deduced

$$\partial_j v_{p\bar{l}}\eta^p = g_{i\bar{l}}\frac{\partial N^i_j(t)}{\partial t}|_{t=0} + v_{i\bar{l}}N^i_j. \tag{15}$$

Replacing this expression in (13), and keeping in mind (3), after an elementary calculation (14) is found. □

Now we are able to construct the adapted frame in respect to the Chern-Finsler (*c.n.c.*) from $(M, \tilde{L}_t)$.

Lemma 3.1. *The adapted frame of the Chern-Finsler (c.n.c) $\tilde{N}(t)$ is $\{\tilde{\delta}_m(t), \dot{\partial}_m, \tilde{\delta}_{\bar{m}}(t), \dot{\partial}_{\bar{m}}\}$, with:*

$$\tilde{\delta}_m(t) = \delta_m - \frac{t}{1+t^2}(\overset{V}{\delta}_m - \partial_m) + \frac{t^2}{1+t^2}\frac{\partial N^k_m(t)}{\partial t}|_{t=0}v^p_k\dot{\partial}_p,$$

and $\tilde{\delta}_{\bar{m}}(t) = \overline{\tilde{\delta}_m(t)}$, where δ_m is the adapted horizontal frame associated to N^i_j from (M, L), and $\overset{V}{\delta}_m$ is its infinitesimal deformation.

With all of the necessary objects, we can build the Chern-Finsler (*c.l.c.*) $\tilde{D}_t = (\tilde{N}^i_j(t),\ \tilde{L}^i_{jk}(t),\ \tilde{C}^i_{jk}(t), 0, 0)$.

Proposition 3.2. *In the complex Finsler space $(M, \tilde{L}_t)$, with the $\tilde{L}_t$ metric from (10), the non-zero local coefficients of the Chern-Finsler (c.l.c.) $\tilde{D}_t$ are:*

$$\begin{aligned}
\tilde{L}^i_{jk}(t) &= \frac{1+2t^2}{1+t^2}L^i_{jk} + \frac{t}{1+t^2}v^i_m L^m_{jk} - \frac{t^2}{1+t^2}\dot{\partial}_k v^i_p\frac{\partial N^p_j(t)}{\partial t}|_{t=0},\\
\tilde{C}^i_{jk}(t) &= \frac{1}{1+t^2}C^i_{jk} + \frac{t}{1+t^2}v^i_m C^m_{jk}.
\end{aligned}$$

The torsion of the Chern-Finsler $N-$ (*c.l.c.*) $\tilde{D}_t$ has the following nonvanishing local coefficients:

$$\begin{aligned}
\tilde{T}^i_{jk}(t) &= \tilde{L}^i_{jk}(t) - \tilde{L}^i_{kj}(t) = \frac{1+2t^2}{1+t^2} T^i_{jk} + \frac{t}{1+t^2} v^i_m T^m_{jk} - \\
&- \frac{t^2}{1+t^2} \left(\dot{\partial}_k v^i_p \frac{\partial N^p_j(t)}{\partial t}|_{t=0} - \dot{\partial}_j v^i_p \frac{\partial N^p_k(t)}{\partial t}|_{t=0} \right), \qquad (16) \\
\tilde{Q}^i_{jk}(t) &= \tilde{C}^i_{jk}(t) = \frac{1}{1+t^2} C^i_{jk} + \frac{t}{1+t^2} v^i_m C^m_{jk}, \\
\tilde{\rho}^i_{j\bar{k}}(t) &= \dot{\partial}_{\bar{k}} \tilde{N}^i_j(t) = \rho^i_{j\bar{k}} + \frac{t}{1+t^2} \frac{\partial \rho^i_{j\bar{k}}(t)}{\partial t}|_{t=0} - \\
&- \frac{t^2}{1+t^2} \left(\dot{\partial}_{\bar{k}} v^i_p \frac{\partial N^p_j(t)}{\partial t}|_{t=0} + v^i_p \frac{\partial \rho^p_{j\bar{k}}(t)}{\partial t}|_{t=0} \right), \\
\tilde{\Theta}^i_{j\bar{k}}(t) &= \delta_{t\bar{k}} \tilde{N}^i_j(t) = \Theta^i_{j\bar{k}} + \frac{t}{1+t^2} \left[\delta_{\bar{k}} \left(\frac{\partial N^p_j(t)}{\partial t}|_{t=0} \right) (\delta^i_p - t v^i_p) - \right. \\
&\left. - \frac{\partial N^p_j(t)}{\partial t}|_{t=0} t \delta_{\bar{k}} v^i_p + \frac{\partial N^{\bar{l}}_{\bar{k}}(t)}{\partial t}|_{t=0} (\delta^{\bar{r}}_{\bar{l}} - t v^{\bar{r}}_{\bar{l}}) \tilde{\rho}^i_{j\bar{r}}(t) \right].
\end{aligned}$$

Theorem 3.3. *Let (M, L) be a complex Finsler space with the infinitesimal deformation V which satisfies the condition $D'V = 0$, where D is the Chern-Finsler (c.l.c.) in (M, L). Then the $\tilde{D}_t = (\tilde{N}^i_j(t), \tilde{L}^i_{jk}(t), \tilde{C}^i_{jk}(t), 0, 0)$ connection of the $(M, \tilde{L}_t)$ space is independent on t.*

Proof. Let us first examine $D'V = 0$ on (M, L).

$$D'v = 0 \;\Rightarrow\; d'v_{i\bar{j}} - \omega^m_i v_{m\bar{j}} = 0.$$

Contracting this relation with $v^{\bar{j}q}$ we obtain $w^{\bar{j}q} d' w_{i\bar{j}} = g^{\bar{j}q} d' g_{i\bar{j}}$. Combining the formula of d' with the expressions of the adapted horizontal frame δ_k and of the vertical co-frame $\delta\eta^k$ in this relation, we can affirm:

$$\overset{CFV}{N^q_k} = v^{\bar{j}q} \frac{\partial v_{i\bar{j}}}{\partial z^k} \eta^i = g^{\bar{j}q} \frac{\partial g_{i\bar{j}}}{\partial z^k} \eta^i = N^q_k. \qquad (17)$$

This gives $d' = \overset{CFV}{d'}$. By the same condition $D'V = 0$, we have

$$\omega^j_k = g^{\bar{m}j} d' g_{k\bar{m}} = v^{\bar{m}j} \overset{CFV}{d'} v_{k\bar{m}} = \overset{CFV}{\omega^j_k}.$$

We check at ones that $D = \overset{CFV}{D}$. □

Remark 3.1. *Using previous Theorem, we can assert that any complex Finsler structure* (M, L) *with infinitesimal deformation* V *which satisfies* $D'V = 0$ *is a rigid one.*

In the following, we will be concerned with the study of particular classes of the first order deformation $\tilde{L}_t = L + tV$.

The notion of the purely Hermitian space ([1, 9]) is presented in the introductory part, and is related to the complex Finsler spaces whose metric $g_{i\bar{j}}$ is independent on η. Supposing that $\tilde{L}_t$ is defining a complex Finsler metric (Th. 2.1) and its infinitesimal deformation V depends only of the position z. In this way we obtain from (11) immediately:

Proposition 3.3. *The complex Finsler space* $(M, \tilde{L}_t)$ *is purely Hermitian if and only if* (M, L) *is purely Hermitian.*

From the expression of the $h-$torsion $\tilde{T}^i_{jk}(t)$ in (16) the following property is deduced:

Proposition 3.4. *Let* (M, L) *be a complex Kähler space.* $(M, \tilde{L}_t)$ *is a complex Kähler space if and only if*

$$\dot{\partial}_k v^i_p \frac{\partial N^p_j(t)}{\partial t}|_{t=0} - \dot{\partial}_j v^i_p \frac{\partial N^p_k(t)}{\partial t}|_{t=0} = 0.$$

From [5] we know that if $\dot{\partial}_{\bar{h}} G^i = 0$ than the space is generalized Berwald, and if in addition the space is Kähler, than it becomes a complex Berwald one.

The complex spray derived from the $(c.n.c.)$ $\tilde{N}^i_j(t)$ is

$$\tilde{G}^i(t) = G^i + \frac{t}{1+t^2} \frac{\partial G^i(t)}{\partial t}|_{t=0} \left(\delta^i_p - t v^i_p\right). \tag{18}$$

In the following, we give necessary and sufficient conditions so that the complex Finsler space $(M, \tilde{L}_t)$ became a generalized Berwald one, or a complex Berwald one.

Proposition 3.5. *Let* (M, L) *be a generalized Berwald space. . Under one of the condition sets stated below* $(M, \tilde{L}_t)$ *is a generalized Berwald space:*

i) the tensors v^i_j *and* $g^{\bar{m}i} \partial_0 v_{0\bar{m}}$ *are holomorphic;*

ii) (M, V) *is generalized Berwald space.*

Proof. Let (M, L) is a generalized Berwald space. From the expression of the complex spray $\tilde{G}^i$, given in (18), is deduced that the spray is holomorphic if and only if $\frac{\partial}{\partial\bar{\eta}^h}\left(\frac{\partial G^i(t)}{\partial t}|_{t=0}\right) = 0$, where $\frac{\partial G^i(t)}{\partial t}|_{t=0} = \frac{\partial N^i_j(t)\eta^j}{\partial t}|_{t=0} = \overset{V}{N^i_j}\,\eta^j$ with $\overset{V}{N^i_j}$ given in (6):

$$\frac{\partial}{\partial\bar{\eta}^h}\left(\frac{\partial G^i(t)}{\partial t}|_{t=0}\right) = (\,\dot{\partial}_{\bar{h}}v^{\bar{m}i}\;\partial_j g_{pm} + v^{\bar{m}i}\;\dot{\partial}_{\bar{h}}\partial_j g_{p\bar{m}} + \dot{\partial}_{\bar{h}}g^{\bar{m}i}\;\partial_j v_{p\bar{m}} + g^{\bar{m}i}\;\dot{\partial}_{\bar{h}}\partial_j v_{\bar{p}m}\,)\,\eta^p\eta^j. \qquad (19)$$

To prove $i)$ we have to go through the following steps. We explicit the equality $\dot{\partial}_{\bar{h}}G^i = 0$ in the following way:

$$\dot{\partial}_{\bar{h}}\left(g^{\bar{m}i}\partial_j g_{p\bar{m}}\eta^i\eta^p\right) = 0 \Leftrightarrow \left(\dot{\partial}_{\bar{h}}g^{\bar{m}i}\partial_j g_{p\bar{m}} + g^{\bar{m}i}\dot{\partial}_{\bar{h}}\partial_j g_{p\bar{m}}\right)\eta^p\eta^j = 0.$$

Contracting this relation with $g_{i\bar{k}}$ we obtain

$$\dot{\partial}_{\bar{h}}\partial_j g_{p\bar{k}}\eta^p\eta^j = -g_{i\bar{k}}\dot{\partial}_{\bar{h}}g^{\bar{m}i}\partial_j g_{p\bar{m}}\eta^p\eta^j. \qquad (20)$$

Assuming $\dot{\partial}_{\bar{h}}v^i_p = 0$ leads us to

$$\dot{\partial}_{\bar{h}}v^{\bar{m}i}g_{p\bar{m}} = -v^{\bar{m}i}\dot{\partial}_{\bar{h}}g_{p\bar{m}} \Leftrightarrow \dot{\partial}_{\bar{h}}v^{\bar{k}i} = -v^{\bar{m}i}g^{\bar{k}p}\dot{\partial}_{\bar{h}}g_{p\bar{m}}. \qquad (21)$$

Replacing (20) and (21) in (19), and using the property $\dot{\partial}_{\bar{h}}\delta^i_j = 0$, we get

$$\frac{\partial}{\partial\bar{\eta}^h}\left(\frac{\partial G^i(t)}{\partial t}|_{t=0}\right) = \left(\dot{\partial}_{\bar{h}}g^{\bar{m}i}\partial_j v_{p\bar{m}} + g^{\bar{m}i}\dot{\partial}_{\bar{h}}\partial_j v_{p\bar{m}}\right)\eta^p\eta^j = \dot{\partial}_{\bar{h}}\left(g^{\bar{m}i}\partial_j v_{p\bar{m}}\right)\eta^p\eta^j.$$

Therefore, if we impose that the tensors v^i_p and $g^{\bar{m}i}\partial_0 v_{0\bar{m}}$ to be holomorphic, then $(M, \tilde{L}_t)$ becomes a generalized Berwald space.

The $ii)$ affirmation is obtained immediately from the expression of $\tilde{G}^i$ from (18) applying the definition of a generalized Berwald space:

$$\frac{\partial}{\partial\bar{\eta}^h}\left(\frac{\partial G^i(t)}{\partial t}|_{t=0}\right) = \frac{\partial}{\partial\bar{\eta}^h}\left(\frac{\partial N^i_j(t)\eta^j}{\partial t}|_{t=0}\right) = \dot{\partial}_{\bar{h}}\;\overset{V}{N^i_j}\;\eta^j = 0.$$

□

Proposition 3.6. *Let (M, L) be a complex Berwald space. Under one of the condition sets stated below $(M, \tilde{L}_t)$ is a complex Berwald space:*

i) the tensors v^i_j, $g^{\bar{m}i}\partial_0 v_{0\bar{m}}$ *are holomorphic, and* $\dot{\partial}_0 v^i_p \frac{\partial N^p_k(t)}{\partial t}|_{t=0} = 0$.

ii) (M, V) *is a complex Berwald space.*

Proof. $(M, \tilde{L}_t)$ is a complex Berwald space, according to [**?**], if and only if it is a generalized Berwald space and a Kähler space, namely the Propositions 3.5 and 3.4 are verified. □

Further, we analyze under what conditions the complex Finsler space $(M, \tilde{L}_t)$ and (M, L) are projectively related, i.e. they have the same geodesics as sets of points.

Proposition 3.7. *The Finsler functions* L *are* $\tilde{L}_t$ *are projectively related if and only if, the complex spray* $\frac{\partial G^i(t)}{\partial t}|_{t=0}$ *of* (M, V) *is independent from* t. *In this case, the projective change is* $\tilde{G}^i = G^i$.

Proof. From the Theorem 1.2 we deduce that, two complex Finsler functions L and $\tilde{L}_t$ are in projective relation if an only if they verify the relation:

$$\dot{\partial}_{\bar{r}}(\delta_k \tilde{L}_t)\eta^k + 2(\dot{\partial}_{\bar{r}} G^l)(\dot{\partial}_l \tilde{L}_t) = \frac{1}{\tilde{L}_t}(\delta_k \tilde{L}_t)\eta^k(\dot{\partial}_{\bar{r}}\tilde{L}_t). \tag{22}$$

By the lemma below, we can express the terms of (22).

Lemma 3.2 ([4])**.** *Let* (M, L) *be a complex Finsler space and* $^*L$ *a complex Finsler metric on* M. *The spray coefficients* G^i *and* $^*G^i$ *of the metrics* L *and* $^*L$ *satisfy:*

$$^*G^i = G^i + \ ^*g^{\bar{r}i}\left(\dot{\partial}_{\bar{r}}(\delta^*_k L)\eta^k + 2(\dot{\partial}_{\bar{r}} G^l)(\dot{\partial}^*_l L)\right) \quad i = 1, \ldots, n. \tag{23}$$

From (18) and (23) we obtain

$$\dot{\partial}_{\bar{r}}(\delta_k \tilde{L}_t)\eta^k + 2(\dot{\partial}_{\bar{r}} G^l)(\dot{\partial}_l \tilde{L}_t) = t g_{p\bar{r}} \frac{\partial G^p(t)}{\partial t}|_{t=0}. \tag{24}$$

Contracting this relation with $\bar{\eta}^r$, and using the homogeneity property of $\tilde{L}_t$, we find:

$$(\delta_k \tilde{L}_t)\eta^k = 2t\dot{\partial}_p L \frac{\partial G^p(t)}{\partial t}|_{t=0}. \tag{25}$$

Replacing the expressions (24) and (25) in (23) we get:

$$\frac{\partial G^p(t)}{\partial t}|_{t=0}\left(g_{p\bar{r}}\tilde{L}_t - \dot{\partial}_p L\ \dot{\partial}_{\bar{r}}\tilde{L}_t\right) = 0.$$

In the above relation the expression in the brackets in generally is not zero, but the infinitesimal deformation of the complex spray G^p is vanishing if the spray is independent from t.

The converse implication is obtained immediately from the formula (18). □

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